Homework 5

(Due Thursday, March 21)

Part 1

If we discount migration to and from Boston, the rate of change of the population is proportional to the population. In particular, each resident of Boston creates an average of one baby every sixty years. (This is more reasonable than it sounds – when a child is born, each parent only creates half a baby!)

(1) Using only this information, make a discrete model by writing a yearly updating function for Boston's population in the form $B_{t+1} = f(B_t) = ...$
(2) Sketch your updating function.
(3) Make a continuous model by writing a formula for the population derivative in terms of the population, in the form $\frac{\partial}{\partial t}B = F(B) = ...$
(4) Sketch the graph of $\frac{\partial}{\partial t}B$ vs. B, and draw a phase line.

Boston residents live about eighty years, so on average one eightieth of the population dies per year.

(5) Revise both models and draw new sketches to take this information into account.

When there are fewer than 600,000 residents, the job prospects are good, and new residents move into the city at an average rate of 5,000 per year. When the population is above 600,000, residents leave to look for work elsewhere at an average rate of 5,000 per year.

(6) Revise your discrete model and draw a new sketch to include this information. Describe the dynamics.
(7) Now revise your continuous model to include this information. Is there a fixed point? Why is this question a little tricky to answer?
(8) Can you explain why the discrete and continuous dynamics are different?

A slightly more realistic model might say that the rate per year at which new residents arrive is $1.5(600,000 - B)$. (When it is negative, residents are leaving.)

(9) Add this information to your continuous model. Where is the fixed point?

As it turns out, all of the new Boston residents are coming from Providence, RI, and every resident who leaves Boston moves to Providence. (This is obviously ridiculous – why would anyone move from Boston to Providence? But work with it anyways.)

(10) Plot the two populations on a phase plane. (For this problem, ignore the birth and death rates, and just consider migration.) Where are the nullclines? Where are the fixed points? Why are there so many?
(11) Describe a situation in which the Boston and Providence nullclines might cross at a single point. Include enough information to explain the behavior in each quadrant.

Part 2

The rate of synthesis of protein P in a cell is directly proportional to the concentration of the mRNA that codes for it within that cell. Let p be the concentration of the protein, and let m be the concentration of the corresponding mRNA. Protein is produced at rate rm.

(1) If initially $p = 0$ and m is fixed at 1, what is the solution for protein concentration over time? (Assume the protein does not degrade.)

Protein P degrades at rate $-\lambda p$, where $\lambda > 0$.

(2) If initially $p = 1$ and there is no mRNA present, what is the solution for protein concentration over time?

The derivative of protein concentration is the sum of the synthesis and the degradation.

$$\dot{p} = rm - \lambda p.$$

(3) If m is fixed at 1, what is the fixed point for p? Is it stable?
(4) Draw the m vs. p phase plane. Assuming $\dot{m} = 0$, sketch the vector field and draw the p-nullcline.

The rate of production of the mRNA coding for protein P is a function of the concentration of that protein in the cell. The effect of p on $\dot{m}$ is modeled by

$$f(p) = \frac{k}{1 + p^n}$$

where $k > 0$ and $n \geq 2$.

(5) Assuming $k = 1$, sketch this function for $n = 2$ and $n = 4$. What might it look like for $n = 1000$? For what values of p is mRNA production "on," and for what values is it "off?"
(6) If p is fixed at zero, what happens to m? If p is fixed at a number much greater than 1, what happens to m?

The mRNA decays at rate γ, so the differential equation describing it is

$$\dot{m} = f(p) - \gamma m$$

(7) Assuming $\dot{p} = 0$, sketch a vector field on the phase plane and draw the m-nullcline.

Put together, we have a system of two equations:

$$\begin{cases} \dot{p} = rm - \lambda p \\ \dot{m} = f(p) - \gamma m \end{cases}$$

(8) On the phase plane, draw both nullclines, cross-hatch them appropriately, and label each region of the phase plane with a vector pointing in the appropriate direction. How many equilibria are there, and do any look stable?
(9) What is a possible behavior of this system?

The increased presence of a protein slows its own production. This is called "negative feedback." The mechanisms that enable homeostasis are all variations on this process.

(10) What is one biological system with negative feedback? Explain the relationship between the state variables.

Part 3

Repeat Part 2 from question (5) to question (9) using $f(p) = \frac{kp^n}{1+p^n}$.

In this case, the presence of small amounts of protein slows its production, while the presence of large amounts of the protein speeds its own production until the concentration reaches some maximal value. This is called "positive feedback," and is important for biological processes of differentiation, in which a system must choose one of multiple distinct attracting states.

(10) What is one biological system with positive feedback? Explain the relationship between the state variables.

Part 4

Consider the Fitzhugh-Nagumo neuron model presented in class, with the addition of a constant inward current I:

$$\begin{cases} \dot{V} = -V(a-V)(1-V) - w + I \\ \dot{w} = \epsilon(V - \gamma w) \end{cases}$$

(1) On a phase plane, draw the nullclines for $I = 0$. Draw them again on the same phase plane for $I > 0$. How does the equilibrium change as I increases?

(2) What qualitative change occurs in the behavior as I increases?

As we will discuss in class, it is often helpful to think of ϵ as "vanishingly small". In this approximation, changes in the V direction occur so much faster than changes in the w direction that we can assume w is fixed while V is changing, and w only changes when V has reached a stable fixed point.

(3) Using the assumption of "vanishingly small" ϵ, what is the crucial event that changes the qualitative behavior of this system?

(4) Suppose that a large current I was only active for a short time, after which it returned to 0. Using the two different V nullclines, explain why a short pulse of current could lead to a spike, while a *very* short pulse might not.

Part 5 Do the following additional problems:

- Chapter 3, Supplementary Problem 28 (A type of butterfly...).
- Chapter 3, Supplementary Problem 29 (A population of size x_t follows the rule...). When it asks for the behavior of the "approximate dynamical system," describe it quantitatively. Is this an example of positive or negative feedback?
- Chapter 5, Supplementary Problems 1-3. (Consider the differential euqation $\frac{dC}{dt}$...)

Part 6 What are you thinking of doing your final project on?

If anything on this assignment looks like it might be a typo, do not hesitate to email me about it.