How to Analyze a 2D Continuous Time Dynamical System

This handout contains the ten steps needed to take to draw and analyze the phase plane of the 2D continuous time dynamical system:

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) = ... \\ \dfrac{dy}{dt} = G(x,y) = ... \end{cases}$$

As you read this, follow along with the step-by-step example on the following pages.

(1) Draw a phase plane with x on the x-axis and y on the y-axis.

- Should you include negative values of x and y? See if negative values make sense for the specific problem.

(2) Set $\frac{dx}{dt} = F(x,y) = 0$ and graph the resulting relation. This relation describes the x-nullcline.

- The equation *might not be a function of* y *in terms of* x: it is just a relation between x and y. For instance, the x-nullcline might be described by the equation $0 = y^2 - x + 1$. It may be easier to graph x as a function of y: $x = y^2 + 1$. Graph it however you can!

(3) Crosshatch the x-nullcline with vertical lines. (On the x-nullcline, there is no motion to the left or right, so any motion must be vertical.)

(4) Put some right arrows on the side of the x-nullcline where $\frac{dx}{dy} = F(x,y)$ is positive, and some left arrows where it is negative.

- You can figure out which side is which by calculating $F(x,y)$ at one point on each side.
- If we ignore the second differential equation and set $\frac{dy}{dt} = 0$, then these arrows describe the whole system. In this case only x is changing, so all motion is horizontal. At any value of y, a horizontal phase line describes the motion of x.

(5) Set $\frac{dy}{dt} = G(x,y) = 0$ and graph the resulting relation. This relation describes the y-nullcline. (Again, y may not be a function of x).

(6) Crosshatch the y-nullcline horizontally. (No motion can occur in the y direction, so it's all right or left.)

(7) Attach vertical arrows to the arrows you drew for (4). Put upward arrows on the side of the nullcline where $\frac{dy}{dt} = G(x,y)$ is positive and downward arrows where it's negative.

- These arrows describe the system if we ignore the first differential equation and set $\frac{dx}{dt} = 0$.

(8) In each region, combine the vertical and horizontal arrows into diagonal arrows.

(9) Draw a trajectory by following the arrows! The trajectory can only cross nullclines in the direction of the crosshatching.

(10) *Check to make sure the arrows make sense.* You should be able to describe in words what is happening in each region.

Example:

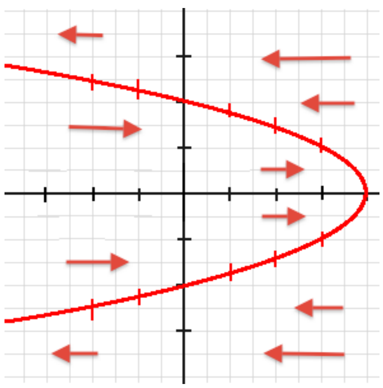
$$\begin{cases} \dfrac{dx}{dt} = F(x,y) = -y^2 - x + 4 \\ \dfrac{dy}{dt} = G(x,y) = 3x - y \end{cases}$$

(1) We know nothing about the problem yet, so we will include negative x and y values.

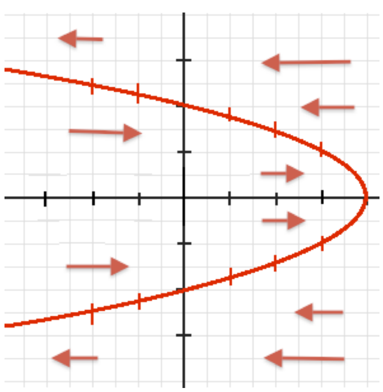
(2) The x-nullcline is described by the relation:

$$\begin{aligned} 0 = \frac{dx}{dt} = F(x,y) =& -y^2 - x + 4 \\ x =& -y^2 - 4 \end{aligned}$$

(3) Red is the x-nullcline, appropriately crosshatched to prevent horizontal motion.



(4) At $(0,0)$, $\frac{dx}{dt} = 4 > 0$. At $(0,3)$, $\frac{dx}{dt} = -5 < 0$. We conclude that right arrows go on the right side of the nullcline and left arrows on the left side.



If we fix $y = 2$, the dynamics of x are described by the equation

$$\frac{dx}{dt} = F(x,2) = -2^2 - x + 4 = -x$$

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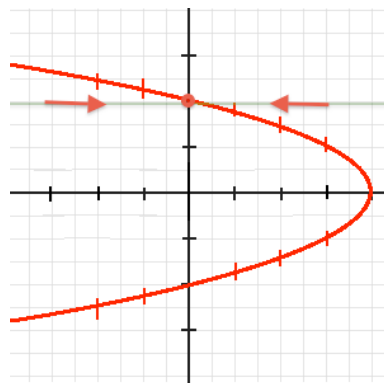
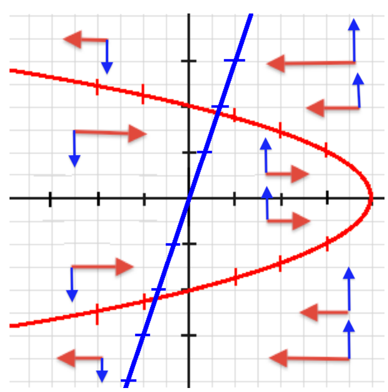


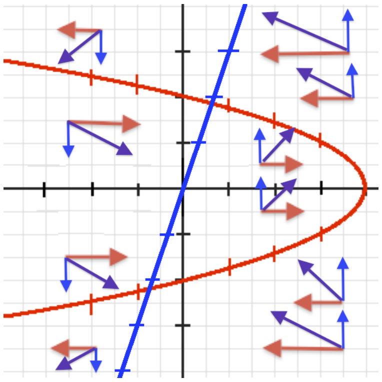
(5) The y-nullcline is described by the relation:

$$\begin{aligned} 0 = \frac{dy}{dt} = G(x,y) =& 3x - y \\ y =& 3x \end{aligned}$$

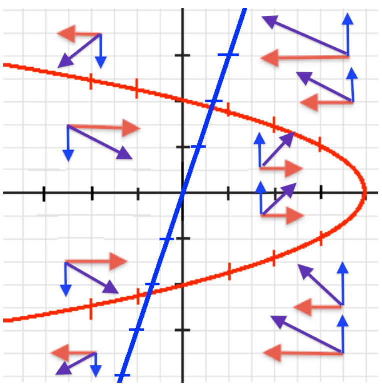
(6) Blue is the y-nullcline, appropriately crosshatched to prevent vertical motion.



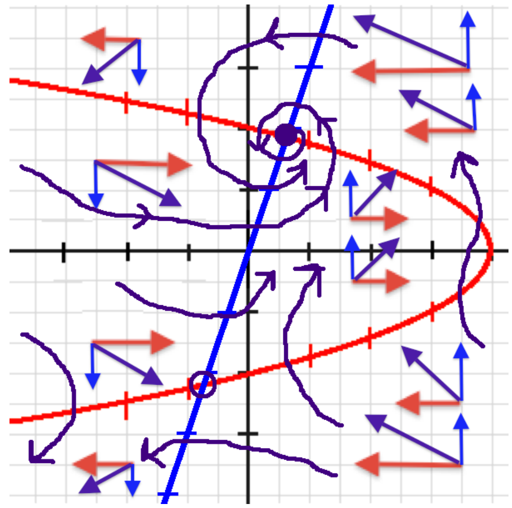
(7) At $(0, 3)$, $\frac{dy}{dt} = -3 > 0$. At $(0, -3)$, $\frac{dy}{dt} = 3 > 0$. We conclude that upward arrows go on the left of the y-nullcline and downward arrows on the left.



(8) Combining vertical and horizontal arrows...



(9) A few trajectories might look like this:



(10) Given an application, we could check our work by trying to explain why arrows point up when x is very large, right when x is small or negative and y is small... etc.